8.6. Differential Forms

Part II

§ 8.6. Differential forms on shapes

A. Motivation: Organize calculation of integrals in various dimensions, so that the dizzing variety of integrals & of versions of the Fundamental theorem of calculus makes sense.

B. On any shape S we will consider $A(S)$

$A^0(S) =$ 0-forms = functions $= C^\infty(S)$.
$A^1(S) =$ 1-forms = things like $f(x)\,dx$
$A^2(S)$ 2-forms = thing like $f\,dx\,dy$
$A^3(S) =$ 3-forms = things like $f\,dx\,dy\,dz$

and

k-forms for all $k = 0, 1, 2, 3, \ldots, \dim(S)$

Remark. If $\dim(S) = n$ then for $k > n$ all k-forms on S are zero!

C. Example: On $S = \mathbb{R}^3$ x, y, z coords.

$A^0(\mathbb{R}^3) = C^\infty(\mathbb{R}^3) =$ functions $f(x,y,z)$ which are ∞-differentiable.

symbol

$A^1(\mathbb{R}^3) =$ expressions $P\,dx + Q\,dy + R\,dz$ where P, Q, R are functions

$d^2x,\ d^2xz$ $\quad dx, dxz$ $\quad dxz\,dxzz$

$A^2(\mathbb{R}^3) =$ expressions $A\,dx\,dy + B\,dx\,dz + C\,dy\,dz$ where A, B, C are functions

$A^3(\mathbb{R}^3) =$ expressions $f\,dx\,dy\,dz$ where f is a function.

Requirement 2.

orientations!

The notion of k-forms as "things ω that can be integrated over oriented k-dimensional subshapes $T \subseteq S$."

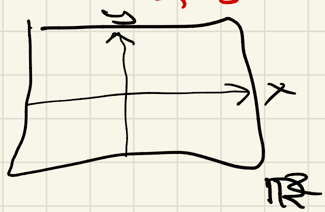
I should take into account the orientation of T.

If T^- is T with opposite orientation then $\int_{T^-}\omega = -\int_{T}\omega$.

Consequences for forms on $\mathbb{R}^n$:

Example: Consider $S = \mathbb{R}^2 \subseteq \mathbb{R}^2_{x,y}$.

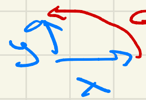
This is a plane:



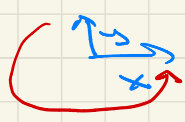
and it has two orientations:

① 

can be described by "x before y":



②  is "y before x".



The 1st orientation will correspond to the basic 2-form $dxdy$.

The 2nd should correspond to "$dydx$"

but:

we have not defined $dx_2\,dx_1$ as a basic 2-form!

This requires a refinement of the symbol $dx\,dy$ to: $dx \wedge dy$, called (exterior) product of 1-forms with the properties:

- $dx \wedge dy = dx\,dy$ — relation of new & old symbol.
- $dy \wedge dx = -dx \wedge dy$.

since changing the order of x, y corresponds geometrically to changing orientation of the plane and that should reverse the integrals:

$$\iint f\, dy \wedge dx = - \iint f\, dx \wedge dy.$$

Consequences:

of our definitions:

(1) k forms on $\mathbb{R}^n$ are of the form

$$\sum_{i} f_{i_1<\dots<i_k}\, dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k} = \omega$$

|| refined

$dx_{i_1}\ dx_{i_2} \cdots dx_{i_k}$
(basic k-forms).

(2) When switching places in wedge a minus appears:

$$\alpha \wedge \beta \wedge \delta = (-1)^{7\cdot 5}\, \alpha \wedge \delta \wedge \beta$$

α: 2-form, β: 7-form, δ: 5-form

{If $\beta = dx$ & $\delta = dy$ get (-1)!!}

switch 7 1-forms with 5 1-forms
get (-1) $7\cdot 5$ times.

(3) There is an operation called (de Rham) differential $d : A^k(S) \to A^{k+1}(S)$

$k = 0$ $\quad d : A^0(\mathbb{R}) \longrightarrow A^1(\mathbb{R})$

$\qquad\qquad\quad C^\infty(\mathbb{R}) \qquad\quad C^\infty(\mathbb{R})\, dx$

$$df = f'(x)\, dx.$$

The End:

E0. $\int_{\text{point } p} f = f(p)$

E1. $\int_C F \cdot dr = \int_C \overbrace{P\,dx + Q\,dy + R\,dz}^{1\text{-form}}$

C a curve in $\mathbb{R}^3$, $F = \langle P, Q, R \rangle$

Integrals of vector fields over curves are just integrals of 1-forms!

E2. $\int_S F \cdot d\vec{S}$ here $F = \langle P, Q, R\rangle$

$\quad ||$

$\int_S P\,\underbrace{dx\,dy}_{} + Q\,dy\,dz + R\,dz\,dx$

$\int_S$ $dx \wedge dy$ everywhere

Integrals of vector fields over surfaces are really integrals of 2-forms!

---

df = ordinary diff. $= \underbrace{f_x\,dx + f_y\,dy + f_z\,dz}_{1\text{-form!}}$

$d\,(F = \langle P, Q, R\rangle = 2\text{-forms})$
$= \operatorname{curl}(F)$, operations div, curl are special cases

of the de Rham differential

Now Stokes theorem on any S:

FTC

$$\int_S \underline{d}\,\omega = \int_{\partial S} \omega$$

de Rham differential

correct notion of derivative in FTC is de Rham d.

k-form.

Thursday: Final Project!